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FERMION STATES ON THE SPHERE S^2

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We solve for the spectrum and eigenfunctions of Dirac operator on the sphere. The eigenvalues are nonzero whole numbers. The eigenfunctions are two-component spinors which may be classified by representations of the $SU(2)$ group with half-integer angular momenta. They are special linear combinations of conventional spherical spinors.

KEYWORDS:
Dirac operator, spherical spinors, spherical functions, Jacobi polynomials

Introduction

The reasons that inspired us to address the problem were primarily physical. There are at least two fields where the results can be applied. The first is the bag model with spectral boundary conditions[1]. This may be of interest for physics of strong interactions. On the other hand it is known that electrons in extensively studied currently fulleren molecules (such as C_{60} and others) obey the Dirac equation. Our problem bears the direct relation to the continuous limit of electronic states in fullerens.

Surprisingly but despite all its beauty the problem escaped textbooks. A general construction of the eigenfunctions of Dirac operator on N-dimensional spheres was given in paper[2]. It was shown that those can be written in terms of Jacobi polynomials. Our present goal was to study in detail and find explicit formulas for the particular case and to emphasize the group properties of solutions. We shall start from approaching the eigenvalue problem in Sect. 2 and construct the $SU(2)$-algebra in Sect. 3. This makes possible to classify the obtained in Sect. 4 solutions under the group representations. Finally after making a bridge to conventional spherical spinors in Sect. 5 we shall summarize the results at the end.

1. The Dirac operator

First of all we shall introduce the notation and then write down the Dirac operator. We shall use the standard parameterization of the unit sphere S^2:

$$x = \sin\theta\cos\phi; \quad y = \sin\theta\sin\phi; \quad z = \cos\theta; \tag{1}$$

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Spinors in two dimensions have two components and the role of Dirac matrices belongs to Pauli matrices: $(\gamma^1, \gamma^2) \to (\sigma_x, \sigma_y)$. Dirac operator is a convolution of covariant derivatives in spinor representation with zweibein $e^{\alpha\, a} = \mathrm{diag}(1, \sin^{-1}\theta)$ and σ-matrices:

$$-i\hat{\nabla} = -i\, e^{\alpha\, a}\sigma_a \nabla_\alpha = -i\sigma_x \left(\partial_\theta + \frac{\cot\theta}{2}\right) - \frac{i\sigma_y}{\sin\theta}\,\partial_\phi. \tag{2}$$

The general theorem (which also applies to the sphere) states that Dirac operator has no zero eigenvalues on manifolds with positive curvature. Therefore there must be no zero value in the spectrum of operator (2).

2. The eigenvalue problem

Eigenfunctions of Dirac operator are two-component spinors $\psi_\lambda(\theta, \phi)$ that satisfy the equation:

$$-i\,\hat{\nabla}\psi_\lambda(\theta, \phi) = \lambda\,\psi_\lambda(\theta, \phi). \tag{3}$$

Expanding them into Fourier series $\psi_\lambda(\theta, \phi) = (2\pi)^{-\frac{1}{2}} \sum_m \psi_{\lambda m}(\theta) \exp i\, m\, \phi$ we obtain independent equations for the components. Obviously because of ψ_λ being spinors the sum runs over all half-integer values of m.

The square of $-i\hat{\nabla}$ is a diagonal operator. After the change of variables $x = \cos\theta$, $x \in [-1, 1]$ we obtain separate generalized hypergeometric equations for the upper and lower components $\alpha_{\lambda m}(x)$ and $\beta_{\lambda m}(x)$:

$$\left[\frac{d}{dx}\,(1-x^2)\,\frac{d}{dx} - \frac{m^2 - \sigma_z\, m\, x + \frac{1}{4}}{1-x^2}\right] \begin{pmatrix} \alpha_{\lambda m} \\ \beta_{\lambda m} \end{pmatrix} = -\left(\lambda^2 - \frac{1}{4}\right) \begin{pmatrix} \alpha_{\lambda m} \\ \beta_{\lambda m} \end{pmatrix}. \tag{4}$$

Because of $\sigma_z = \mathrm{diag}(1, -1)$ in the second term the equations for α and β differ. Inversion $x \to -x$ (or, equivalently, $m \to -m$) transforms one into another.

A regular routine brings (4) to equations of hypergeometric type. Those have square integrable on the interval $x \in [-1, 1]$ solutions provided that

$$\lambda^2 = \left(n + |m| + \frac{1}{2}\right)^2, \tag{5}$$

with integer $n \geq 0$. Thus $\lambda = \pm 1, \pm 2 \ldots$ are nonzero integers and indeed there are no zero-modes of Dirac operator on the sphere. The solutions may be expressed in terms of Jacobi polynomials of n-th order, like in [2]. We shall put them into another form that will be given later, see Eqs. (10).

3. The $SU(2)$ algebra

Let us show that Dirac operator on the sphere S^2 is invariant under transformations of the $SU(2)$ group. Its algebra consists of three operators:

$$\hat{L}_z = -i\frac{\partial}{\partial\phi}; \tag{6a}$$

$$\hat{L}_+ = e^{\,i\phi}\left(\frac{\partial}{\partial\theta} + i\,\cot\theta\frac{\partial}{\partial\phi} + \frac{\sigma_z}{2\,\sin\theta}\right); \tag{6b}$$

$$\hat{L}_- = -e^{-i\phi}\left(\frac{\partial}{\partial\theta} - i\,\cot\theta\frac{\partial}{\partial\phi} - \frac{\sigma_z}{2\,\sin\theta}\right). \tag{6c}$$

The operators satisfy the standard commutation relations of $SU(2)$ algebra:

$$\left[\hat{L}_z,\,\hat{L}_+\right] = \hat{L}_+; \qquad \left[\hat{L}_z,\,\hat{L}_-\right] = -\hat{L}_-; \qquad \left[\hat{L}_+,\,\hat{L}_-\right] = 2\hat{L}_z. \tag{7}$$

The direct check proves that they also commute with Dirac operator (2). Hence its eigenfunctions may be classified by representations of the $SU(2)$-group. Action of the generators $\hat{L}_+$ ($\hat{L}_-$) raises (lowers) the value of m leaving λ intact.

The spherical d'Alembert operator is directly linked to the square of angular momentum:

$$-\hat{\nabla}^2 = \hat{L}^2 + \frac{1}{4} = \hat{L}_z^2 + \frac{1}{2}(\hat{L}_+\hat{L}_- + \hat{L}_-\hat{L}_+) + \frac{1}{4}. \tag{8}$$

This results into the following relation between the eigenvalues:

$$\langle m,\,n|\hat{L}^2|m,\,n\rangle = \lambda^2 - \frac{1}{4} = (n+|m|)(n+|m|+1). \tag{9}$$

The proper values of angular momentum are half-integers $l = |\lambda| - \frac{1}{2} = n + |m|$.

Operators (6) are diagonal. Hence formally α and β belong to different representations. Nevertheless they form doublets with respect to the Dirac operator.

4. The spinor spherical functions

We shall list spinor spherical functions $\Upsilon^{\varepsilon}_{lm}$ by the values of angular momentum $l = n+|m|$, its z-projection m and $\varepsilon = \operatorname{sgn}\lambda$. Let us introduce the integers $l^{\pm} = l \pm \frac{1}{2}$ and $m^{\pm} = m \pm \frac{1}{2}$. Using the $\pm$-superscripts for ε we may write:

$$\Upsilon^{\pm}_{lm}(x,\,\phi) = \pm\frac{i^{l^+}(-1)^{l^-}}{2^{l^+}\Gamma(l^+)}\sqrt{\frac{(l+m)!}{(l-m)!}}$$
$$\times \quad \frac{e^{im\phi}}{\sqrt{2\pi}}\begin{pmatrix} \sqrt{\mp i}(1-x)^{-\frac{m^-}{2}}(1+x)^{-\frac{m^+}{2}}\,\frac{d^{l-m}}{dx^{l-m}}(1-x)^{l^-}(1+x)^{l^+} \\ \sqrt{\pm i}(1-x)^{-\frac{m^+}{2}}(1+x)^{-\frac{m^-}{2}}\,\frac{d^{l-m}}{dx^{l-m}}(1-x)^{l^+}(1+x)^{l^-} \end{pmatrix}. \tag{10}$$

This representation of $SU(2)$ multiplets was constructed by successive application of operator $\hat{L}_-$ to the function with maximum projection $m = l$ of momentum. The constants were chosen in order to ensure the correct behaviour of spinors under complex conjugation[3] and fix the right signs of matrix elements,

$$\langle l,\,m-1\,|\hat{L}_-|\,l,\,m\rangle = \langle l,\,m\,|\hat{L}_+|\,l,\,m-1\rangle = \sqrt{(l+m)(l-m+1)}; \tag{11a}$$

$$\langle l,\,m\,|\hat{L}_z|\,l,\,m\rangle = m. \tag{11b}$$

The spinor spherical functions constitute a complete orthonormal system:

$$\langle\Upsilon^{\varepsilon_1}_{l_1m_1}|\Upsilon^{\varepsilon_2}_{l_2m_2}\rangle = \int_0^{2\pi}d\phi\int_0^{\pi}(\Upsilon^{\varepsilon_1}_{l_1m_1})^{\dagger}\Upsilon^{\varepsilon_2}_{l_2m_2}\,\sin\theta\,d\theta = \delta^{\varepsilon_1\varepsilon_2}\,\delta_{l_1l_2}\,\delta_{m_1m_2}. \tag{12}$$

This makes them quite a useful means of harmonic analysis on S^2.

5. Relation to the ordinary spherical spinors

Conventional spherical spinors $\Omega_{j,\,l,\,m}$ are characterized by total angular momentum j, orbital angular momentum l and z-projection of the total momentum m. They also form on S^2 a complete functional system that differs from $\Upsilon^{\varepsilon}_{lm}$. The reason is that Ω-spinors were constructed in the flat $3d$-space using the set of σ-matrices aligned with Cartesian frame whereas our γ-matrices were specific to the curved $2d$-manifold S^2. Certainly the two types of spinors must be interrelated.

Transformation of spinors from spherical to Cartesian coordinates includes multiplication by matrix $V^{\dagger} = \exp -\frac{i\sigma_z}{2}\phi \exp -\frac{i\sigma_y}{2}\theta$. This leads to the relation:

$$V^{\dagger}\,\Upsilon^{\pm}_{l\,m} = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{\frac{l+m}{2l}}\,Y_{l^-m^-} \pm \sqrt{\frac{l-m+1}{2l+2}}\,Y_{l^+m^-} \\ \sqrt{\frac{l-m}{2l}}\,Y_{l^-m^+} \mp \sqrt{\frac{l+m+1}{2l+2}}\,Y_{l^+m^+} \end{pmatrix} = \frac{1}{\sqrt{2}}\left(\Omega_{l,\,l^-,\,m} \mp \Omega_{l,\,l^+,\,m}\right). \tag{13}$$

Thus Υ's are combined of Ω's with the same values of j and m. Therefore our functions have definite values of total angular momentum $\hat{J}$ and its z-projection $\hat{J}_z$:

$$\hat{\mathbf{J}}^2\,\Upsilon_{lm} = (\hat{\mathbf{L}}_C + \hat{\mathbf{S}}_C)^2\,\Upsilon_{lm} = l(l+1)\,\Upsilon_{lm} \qquad \text{and} \qquad \hat{J}_z\,\Upsilon_{lm} = m\,\Upsilon_{lm}; \tag{14}$$

where $\hat{\mathbf{L}}_C$ and $\hat{\mathbf{S}}_C$ are the Cartesian operators of orbital momentum and spin vectors respectively. In the mean time $\Upsilon^{\pm}_{lm}$ do not diagonalize orbital momentum $\hat{l}$.

6. Summary

We have shown that the spectrum of Dirac operator on the sphere consists of nonzero integers. The eigenfunctions are two-component spherical spinors that can be grouped in $SU(2)$ multiplets with half-integer values of total momentum. Our solutions differ from customary spherical spinors obtained in the flat space being the linear combinations of those.

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